# X. *On Hyperjacobian Surfaces and Curves.*

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## § 1. *Introductory.*

IN a paper published in the 'Mathematische Annalen' (vol. iii. p. 459), BRILL has discussed the question of curves having three-point contact with a doubly infinite pencil of curves; and in particular he has investigated some of the properties of the curve passing through all the points of contact with the individual curves of the pencil. In the same Journal (vol. x. p. 221) KREY of Kiel has applied a method similar to that of BRILL with partial success to the question of curves having four-point contact with a triply infinite pencil. Some formulæ, however, given in my paper "On the Sextactic Points of a Plane Curve" (Phil. Trans. 1865, p. 657) have proved to be directly applicable to both questions. An application of them to BRILL's problem will be found in the 'Comptes Rendus' for 1876 (2nd semestre, p. 627), and a solution of KREY's problem in the 'Proceedings of the London Mathematical Society for the same year (vol. viii. p. 29).

The present subject was in the first instance suggested by the foregoing papers; and from one point of view it may be regarded as an attempt to extend the question to the case of surfaces; viz. to determine a curve which shall pass through the points of contact of a given surface U with certain surfaces belonging to a pencil V, and to investigate some of its properties. From a slightly different point of view, however, it may be considered as an extension of two ideas, viz. first, that of the Jacobian surface, or locus of the points whose polar planes with regard to four surfaces meet in a point; and secondly, that of the Jacobian curve, or locus of points whose polar planes with regard to three surfaces have a right line in common. More particularly, commencing with the facts, first, that if a surface of the form $a\phi+b\psi+c\chi$ touch a surface U, the point of contact is a point on the Jacobian, and secondly, that if a surface of the form $a\phi+b\psi$ touch a surface U, the point of contact is a point on the Jacobian curve, I have endeavoured to extend them to higher degrees of contact.

Consider, then, a surface $U=0$ of the degree $n$, and other surfaces $\phi=0$, $\psi=0$, . . all of the degree $m$, where $m$ is in general different from $n$. Let the first differential coefficients of U, $\phi$, $\psi$, . . be thus expressed:

$$\left.\begin{array}{l} (\partial_x,\ \partial_y,\ \partial_z,\ \partial_t)\ U=u,\ v,\ w,\ k, \\ (\partial_x,\ \partial_y,\ \partial_z,\ \partial_t)\ \phi=a,\ b,\ c,\ d, \\ (\partial_x,\ \partial_y,\ \partial_z,\ \partial_t)\ \psi=a',\ b',\ c',\ d', \\ \qquad\qquad\qquad\quad \vdots\quad \vdots\quad \vdots\quad \vdots\quad \vdots \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then, when the number of surfaces is equal to that of the variables, the determinant

$$\left.\begin{matrix} u, & v, & w, & k \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{matrix}\right\} \quad \ldots \ldots \ldots \quad (2)$$

is called the Jacobian of U, $\phi$, $\psi$, $\chi$, and is usually designated by the formula

$$\mathrm{J}\,(\mathrm{U}, \phi, \psi, \chi) \quad \ldots \ldots \ldots \quad (3)$$

If, however, the number of surfaces be less by unity than that of the variables, the first differential coefficients (1) will form the matrix

$$\left.\begin{matrix} u, & v, & w, & k \\ a, & b, & c, & d \\ a', & b', & c', & d' \end{matrix}\right\} \quad \ldots \ldots \ldots \quad (4)$$

from which there may be derived four determinants, whereof, however, two only are independent. Any two of these will represent surfaces, which may be called Jacobian (more strictly Hypojacobian) surfaces; and their intersection, the Jacobian curve of the system. The pair of surfaces, or their curve of intersection, may conveniently be designated by the formula

$$\mathrm{J}((\mathrm{U}, \phi, \psi)) \quad \ldots \ldots \ldots \quad (5)$$

Again, if $\Delta$U, $\Delta'$U, . . be any derivatives of U beyond the first, we may, by means of them, form from the system of surfaces U, $\phi$, $\psi$, . . the following matrix:—

$$\left.\begin{matrix} u, & v, & w, & k, & \Delta \mathrm{U}, & \Delta' \mathrm{U}, & \ldots \\ a, & b, & c, & d, & \Delta \phi, & \Delta' \phi, & \ldots \\ a', & b', & c', & d', & \Delta \psi, & \Delta' \psi, & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}\right\} \quad \ldots \ldots \quad (6)$$

in which it is to be understood that the number of surfaces is such that the number of columns exceeds that of the lines by unity. From this matrix two independent determinants may be formed; and the surfaces which they represent may, in conformity with previous nomenclature, be called the *Hyperjacobian surfaces*, and their intersection the *Hyperjacobian curve* of the system. These will be designated by the formula

$$\mathrm{J}\,((\mathrm{U}, \phi, \psi, \ldots)) \quad \ldots \ldots \ldots \quad (7)$$

The principal properties of the Jacobian surface (2), and of the Jacobian curve (5) are known; and the object of the present paper is to investigate some of the properties of the Hyperjacobian surfaces and curves.

The surfaces and curves (5) are in themselves independent of any particular mode of origination; but they are here considered as arising out of the contact of the surface U

with a surface passing through the intersection of $\phi$ and $\psi$; or, say, with one or more particular surfaces of the singly infinite or *one-fold pencil* $a\phi+b\psi=V$, where a and b are constants. For the equations expressing the condition of contact between U and V will enable us to eliminate the ratio a : b in two ways, and give rise to the curve $J((U, \phi, \psi))=0$. These equations, combined with $U=0$, will suffice to determine the coordinates of the points of contact; and if the values so determined be substituted in the equation $V=0$, the values of a : b, that is, the particular surfaces of the pencil for which such contact obtains, will be found.

Similarly, the Jacobian (2) may be considered as arising out of the contact of the surface U with the doubly infinite or *two-fold pencil*, $V=a\phi+b\psi+c\chi$.

Again, by selecting suitable derivatives of $U, \phi, \psi, ..$ for the terms $\Delta U, \Delta' U, ..$ the Hyperjacobians (6) and (7) may be considered as arising from contact of higher degrees than common (or two-branch) contact between U and some of the surfaces of the pencil $V=a\phi+b\psi+ ..$ And we shall in each such case have three equations, viz. $U=0$, and the equations of the curve, which will give the values of the coordinates of the points of contact. These values, substituted in the equation for V, will determine one of the ratios a : b : .., and thereby a pencil (whose multiplicity is less by unity than that of the given pencil), for which the contact obtains.

The properties here considered are those which appertain to the points, if any, through which all the surfaces pass, or, as they may be termed, the *principal points* of the system; and consist mainly in the nature of the contact of the Hyperjacobian surfaces with the surface U, and the multiplicity of the Hyperjacobian curve at the points in question.

The present investigation extends to the cases of two-branch contact of the given surface with a one-fold and with a two-fold pencil, and of three-branch contact with a four-fold pencil. In the latter case, notice is also made of some properties appertaining to the points, if any, where all the surfaces touch one another, or, as they may be termed, the *secondary points* of the system. In particular, it is shown that, in the case of common, or two-branch contact and a one-fold pencil, the Jacobian curve has a double point at the principal points; while in the case of three-branch contact and a four-fold pencil, the Hyperjacobian curve has a triple point at the same points.

### § 2. *The Jacobian Surfaces and Curve of a one-fold pencil.*

Consider a surface $U=0$ of the degree $n$, and two other surfaces $\phi=0$, $\psi=0$, each of the degree $m$, where $m$ is in general different from $n$; also the one-fold pencil of surfaces

$$V=a\phi+b\psi=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where a and b are constants. If U and V have common, or two-branch contact, we shall have, beside (1), the following conditions, viz.

$$\partial_x V=\theta u, \ \partial_y V=\theta v, \ \partial_z V=\theta w, \ \partial_t V=\theta k, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\theta$ is indeterminate. If for V there be substituted its value given by (1), these equations will serve to eliminate the ratios $\mathrm{a}:\mathrm{b}:\theta$ in two ways; and by this means we shall obtain two equations in $x, y, z, t$, which, being independent of a, b, will hold good for any particular surfaces of the pencil U, and will consequently represent a curve passing through all the points of contact of the surface U with the pencil V. The resultants, combined with the equation $\mathrm{U}=0$, will determine the coordinates of all the points of contact; and the particular surfaces of the pencil which actually touch U will be found by substituting successively the values of $x:y:z:t$, so determined, in the equation (1), and thence deducing the corresponding values of the ratio $\mathrm{a}:\mathrm{b}$.

The two resultants may be represented by the following formulæ:—

$$\left.\begin{array}{r} \mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S}= u,\ v,\ w,\ k = \mathrm{J}((\mathrm{U},\ \phi,\ \psi))=0, \\ a\ ,\ b\ ,\ c\ ,\ d \qquad\qquad\qquad\qquad \\ a',\ b',\ c',\ d' \qquad\qquad\qquad\qquad \end{array}\right\} \quad \ldots\ldots \quad (3)$$

in which P, Q, R, S represent the four determinants which can be formed from the matrix (3) by the omission of each of the four columns in succession. Of these, of course, two only are independent; and they represent, as mentioned in the Introduction, the Jacobian curve of the system $\mathrm{U}, \phi, \psi$. The Jacobian curve presents some peculiarities at the principal points of the system, *i. e.* at the points where $\mathrm{U}=0$, $\phi=0$, $\psi=0$. In order to examine them, it will be convenient to transform the expressions (3) as follows:—

$$\mathrm{Q}z-\mathrm{R}y=\begin{array}{l} -zw,\ k\ ,\ u \\ zc\ ,\ d\ ,\ a \\ zc',\ d',\ a' \end{array} \qquad \begin{array}{l} -k\ ,\ u\ ,\ yv \\ d\ ,\ a\ ,\ yb \\ d',\ a',\ yb' \end{array}$$

$$=\begin{array}{l} yv+zw,\ u\ ,\ k \\ yb+zc\ ,\ a\ ,\ d \\ yb'+zc',\ a',\ d' \end{array} \qquad = \begin{array}{l} u\mathrm{U},\ u,\ k, \\ m\phi,\ a\ ,\ d \\ m\psi,\ a',\ d' \end{array}$$

or, putting $(n-m):m=\mathrm{m}$, the above expression takes the form

$$\mathrm{Q}z-\mathrm{R}y=\begin{array}{l} \mathrm{U}+\mathrm{mU},\ u\ ,\ k \\ \phi, \qquad\quad a\ ,\ d \\ \psi, \qquad\quad a',\ d' \end{array} =\mathrm{A}+\mathrm{mUA},\ \text{suppose.}$$

More generally, let

$$\left.\begin{array}{r} \mathrm{A}_1,\ \mathrm{B}_1,\ \mathrm{C}_1,\ \mathrm{F}_1,\ \mathrm{G}_1,\ \mathrm{H}_1=a\ ,\ b\ ,\ c\ ,\ d\ , \\ a',\ b',\ c',\ d', \end{array}\right\} \quad \ldots\ldots\ldots \quad (4)$$

*i. e.* $\mathrm{A}_1, \mathrm{B}_1, \ldots$ are the determinants formed from the matrix (4) by omitting the columns two and two in the usual order, viz. 2, 3; 3, 1; 1, 2; 1, 4; 2, 4; 3, 4. Also let

$$\left.\begin{array}{r}A, B, C, F, G, H=\overset{*}{U}, u, v, w, k, \\ \phi, a, b, c, d, \\ \psi, a', b', c', d',\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

*i. e.* A, B, . . are the determinants formed from the matrix (5), by retaining the first column (on that account marked by the asterisk *), and omitting the four others, two and two, in the same order as before. We may then form the following system:—

$$\left.\begin{array}{ll}(Qz-Ry) : m=A+mA_1, & (Pt-Sx) : m=F+mF_1, \\ (Rx-Pz) : m=B+mB_1, & (Qt-Sy) : m=G+mG_1, \\ (Py-Qx) : m=C+mC_1, & (Rt-Sz) : m=HmH_1,\end{array}\right\} \quad . \quad . \quad . \qquad (6)$$

any two of which may be regarded as the equations of the curve in question.

By means of these equations it may easily be shown that at the principal points each of the surfaces (6), or say each of the surfaces

$$\left.\begin{array}{l}x, \ y, \ z, \ t=0 \\ P, \ Q, \ R, \ S\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

touches, or has two-branch contact with U. For, on differentiating the first equation of (6), we obtain

$$\partial_x(Qz-Ry) : m=\partial_x A+muA_1+mU\partial_x A_1.$$

But, on reference to (5), it is clear that at the points in question $(\partial_x, \partial_y, \partial_z, \partial_t)A=0$, or more generally

$$(\partial_x, \partial_y, \partial_z, \partial_t)(A, B, C, F, G, H)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Hence

$$\partial_x(Qz-Ry) : u=\partial y(Qz-Ry) : v=\partial_z(Qz-Ry) : w=\partial_t(Qz-Ry) : k=mmA_1;$$

in other words, the surface $Qz-Ry$ touches U at the principal points of the system. Still more generally, we may write, for the whole group, the following formula, viz.

$$\left.\begin{array}{l}(\partial_x, \partial_y, \partial_z, \partial_t)(x, \ y, \ z, \ t) \\ \qquad\qquad\qquad |P, \ Q, \ R, \ S|\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

$$=mm\,(u, v, w, k)(A_1, B_1, C_1, F_1, G_1, H_1),$$

which expresses the fact that, at the principal points of the system, each of the surfaces (7) has a two-branch contact with U. It remains to show that the same is the case with each of the surfaces P, Q, R, S. Since each of the expressions for P, Q, R, S vanishes at the points under consideration, we have

$$\partial_x(Rx-Pz)=x\partial_x R-z\partial_x P=mmB_1u,$$

$$\partial_x(Py-Qx)=y\partial_x P-x\partial_x Q=mmC_1u.$$

But on multiplying the first of these equations by $z$ and the second by $y$, and subtracting, we obtain

$$x(x\partial_x\mathrm{P}+y\partial_x\mathrm{Q}+z\partial_x\mathrm{R})-(x^2+y^2+z^2)\partial_x\mathrm{P}=m\mathrm{m}(z\mathrm{B}_1-y\mathrm{C}_1)u.$$

But, since $x\mathrm{P}+y\mathrm{Q}+z\mathrm{R}=0$, this reduces itself to

$$\partial_x\mathrm{P}:u=m\mathrm{m}(z\mathrm{B}_1-y\mathrm{C}_1):(x^2+y^2+z^2).$$

By a similar process applied to the other equations, and by writing $x^2+y^2+z^2=r^2$, we should obtain

$$\left.\begin{array}{l}\partial_x\mathrm{P}:u=\partial_y\mathrm{P}:v=\partial_z\mathrm{P}:w=\partial_t\mathrm{P}:k=m\mathrm{m}(z\mathrm{B}_1-y\mathrm{C}_1):r^2,\\ \partial_x\mathrm{Q}:u=\partial_y\mathrm{Q}:v=\partial_z\mathrm{Q}:w=\partial_t\mathrm{Q}:k=m\mathrm{m}(x\mathrm{C}_1-z\mathrm{A}_1):r^2,\\ \quad\vdots\qquad\quad\vdots\qquad\quad\vdots\qquad\quad\vdots\qquad\qquad\qquad\vdots\end{array}\right\} \quad . \quad . \quad . \quad (10)$$

Hence each of the surfaces P, Q, R, S has two-branch contact with U at the principal points of the system. In other words, each of the Jacobian surfaces $\mathrm{J}((\mathrm{U}, \phi, \psi))=0$ touches the surface U, and consequently they touch one another at the principal points of the system. At the same points the Jacobian curve therefore has a node, and along each of its two branches the contact of the Jacobian surfaces is three-pointic.

It should here be noticed that if all the surfaces are of the same degree, *i. e.* if $n=m$, then $\mathrm{m}=0$, and consequently

$$(\partial_x, \partial_y, \partial_z, \partial_t)(\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S})=0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

so that, in this case, the Hessian of each Jacobian surface vanishes at the principal points; in other words, the principal points are parabolic points on the Jacobians.

## § 3. *The Hyperjacobian Surfaces and Curve of a two-fold Pencil.*

Consider the surface $\mathrm{U}=0$, as before; the three surfaces $\phi$, $\psi$, $\chi$; and the pencil

$$\mathrm{V}=\mathrm{a}\phi+\mathrm{b}\psi+\mathrm{c}\chi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, it is well known that if two surfaces touch one another, the curve of intersection has a double point at the point of contact, and that along each of the branches the contact is three-pointic. The formulæ for determining the directions of these two branches are as follows. Adopting the notation of my memoir, "On the Contact of Surfaces" (Phil. Trans. 1872, p. 259), in which $\alpha$, $\beta$, $\gamma$, $\delta$, $\alpha'$, $\beta'$, $\gamma'$, $\delta'$, are arbitrary constants, and

$$\begin{aligned}\alpha x+\beta y+\gamma z+\delta t&=\varpi,\\ \alpha' x+\beta' y+\gamma' z+\delta' t&=\varpi',\\ \mathrm{A}&=\alpha'\varpi-\alpha\varpi',\\ \mathrm{B}&=\beta'\varpi-\beta\varpi',\\ \mathrm{C}&=\gamma'\varpi-\gamma\varpi',\\ \mathrm{D}&=\delta'\varpi-\delta\varpi',\end{aligned}$$

$$\left.\begin{array}{l}\mathrm{H}=\begin{array}{lllll} u_1, & w', & v', & l', & \mathrm{A} \\ w', & v_1, & u', & m', & \mathrm{B} \\ v', & u', & w_1, & n', & \mathrm{C} \\ l', & m', & n', & k_1, & \mathrm{D} \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & \mathrm{D}, & . \end{array} \qquad \Delta=\begin{array}{llllll} u_1, & w', & v', & l', & \mathrm{A}, & \partial_x \\ w', & v_1, & u', & m', & \mathrm{B}, & \partial_y \\ v', & u', & w_1, & n', & \mathrm{C}, & \partial_z \\ l', & m', & n', & k_1, & \mathrm{D}, & \partial_t \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & \mathrm{D}, & . & . \\ \partial_x, & \partial_y, & \partial_z, & \partial_t, & . & . \end{array} \\ \mathrm{n}=1+2(m-1):(n-1)\,; \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

then the required formulæ will be

$$\partial_x\mathrm{V} : u=\partial_y\mathrm{V} : v=\partial_z\mathrm{V} : w=\partial_t\mathrm{V} : k=\Delta\mathrm{V} : n\mathrm{H}=\theta, \quad . \quad . \quad . \quad . \quad (3)$$

the last of which, being a quadratic in $\varpi : \varpi'$, will determine the two directions sought.

If the value of V given by the equation (1) be inserted in (3), we may eliminate the ratios $\mathrm{a} : \mathrm{b} : \mathrm{c} : \theta$ in two different ways. The resultants will be the Hyperjacobian surfaces, and their intersection the Hyperjacobian curve.

The two independent results may be comprised in the formula

$$\left.\mathrm{P, Q, R, S, T}=\begin{array}{lllll} u, & v, & w, & k, & \mathrm{nH}=0. \\ a, & b, & c, & d, & \Delta\phi \\ a', & b', & c', & d', & \Delta\psi \\ a'', & b'', & c'', & d'', & \Delta\chi \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Among these, the expressions P, Q, R, S may be combined in the same manner as the corresponding expressions in § 1; and if we then write

$$\left.\mathrm{A, B, C, F, G, H}=\begin{array}{llllll} \overset{*}{\mathrm{U}}, & u, & v, & w, & k, & \overset{*}{\mathrm{n}}\mathrm{H}, \\ \phi, & a, & b, & c, & d, & \Delta\phi, \\ \psi, & a', & b', & c', & d', & \Delta\psi, \\ \chi, & a'', & b'', & c'', & d'', & \Delta\chi, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\left.\mathrm{A_1, B_1, C_1, F_1, G_1, H_1}=\begin{array}{lllll} a, & b, & c, & d, & \overset{*}{\Delta}\phi, \\ a', & b', & c', & d', & \Delta\psi, \\ a'', & b'', & c'', & d'', & \Delta\chi, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we may take instead of the four expressions P, Q, R, S, the six

$$\mathrm{A}+\mathrm{mUA}_1, \ldots \quad \mathrm{F}+\mathrm{mUF}_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and any two of these equated to zero, or any one of them combined with $\mathrm{T}=\mathrm{T}_0+\mathrm{mUT}_1=0$, will serve for the two equations required. From these it is easy to see that, in the same way as in § 1, it may be shown that the Hyperjacobian surfaces touch the surface U at the principal points of the system.

But in this case we may carry the question of the contact of the Hyperjacobians a step further. In fact, bearing in mind that if $p$, $q$ be any two rational, integral, and homogeneous functions of $x, y, z, t$, the nature of the operation $\Delta$ is such that, if we put

$$\Delta=(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N})(\partial_x, \partial_y, \partial_z, \partial_t)^2, \quad . \quad . \quad . \quad . \quad (8)$$

then

$$\Delta pq=p\Delta q+q\Delta p+2(\mathfrak{A}, \mathfrak{B}, \ldots)(\partial_x p, \partial_y p, \partial_z p, \partial_t p)(\partial_x q, \partial_y q, \partial_z q, \partial_t q). \quad . \quad . \quad (9)$$

This being so, if we put

$$\left.\begin{aligned}
\mathfrak{A}\partial_x+\mathfrak{H}\partial_y+\mathfrak{G}\partial_z+\mathfrak{L}\partial_t&=A',\\
\mathfrak{H}\partial_x+\mathfrak{B}\partial_y+\mathfrak{F}\partial_z+\mathfrak{M}\partial_t&=B',\\
\mathfrak{G}\partial_x+\mathfrak{F}\partial_y+\mathfrak{C}\partial_z+\mathfrak{N}\partial_t&=C',\\
\mathfrak{L}\partial_x+\mathfrak{M}\partial_y+\mathfrak{N}\partial_z+\mathfrak{D}\partial_t&=D',
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

it follows that

$$A'U : x=B'U : y=C'U : z=D'U : t=H(n-1). \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If, then, we operate with $\Delta$ upon the equation $(t : m)T=T_0+mUT_1$, and put $\tau_1$ for the degree of $T_1$ in $x, y, z, t$, we shall obtain the following result:—

$$(t : m)\Delta T+2D'T : m=\Delta T_0+4mHT+2m\tau_1 HT_1 : (n-1).$$

Substituting for $D'T$ from the equations $\partial_x T : u=. \,.=mmT_1 : t$, we find

$$(t : m)\Delta T=\Delta T_0+2m\{-1 : (n-1)+2+\tau_1 : (n-1)\}HT_1. \quad . \quad . \quad . \quad (12)$$

But if $\tau$ represent the degree of $T_1$ it is easily seen that $\tau+1=\tau_1+n$, or $2\tau_1+2n-4=2(\tau-1)$; so that the coefficient of $HT_1$ will be $=2+2(\tau-1) : (n-1)$. Again, omitting terms which vanish with $U$, $\phi$, $\psi$, $\chi$, and, for brevity, writing down only the first line of each determinant, we find

$$\begin{aligned}
\Delta T_0= \;& 4H, \ u, \ v, \ w\\
&+2u, \ A'(u, \ v, \ w)\\
&+2v, \ B'(u, \ v, \ w)\\
&+2w, \ C'(u, \ v, \ w)\\
&+2k, \ D'(u, \ v, \ w),
\end{aligned}$$

where the operations $A'$, . . are supposed to affect all the columns which follow them; thus:

$$\begin{aligned}
u, \ A'(u, \ v, \ w)=\;&u, \ A'u, \ v, \ w\\
&+u, \ u, \ A'v, \ w\\
&+u, \ u, \ v, \ A'w.
\end{aligned}$$

This being so,

$$\left.\begin{aligned}
\Delta T_0=\;& 4H, \ u, \ v, \ w\\
&+2D'(k, \ u, \ v, \ w)\\
&-8H, \ u, \ v, \ w\\
=&-4Hu, \ v, \ w\\
&+2D'T.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

But, writing it in full, the first term of this

$$
\begin{array}{llll}
=\mathrm{nH}+(4-\mathrm{n})\mathrm{H}, & u, & v, & w=-\mathrm{T}+(4-\mathrm{n})\mathrm{HT}_1=(4-\mathrm{n})\mathrm{HT}_1, \\
\Delta\phi, & a\,, & b\,, & c \\
\Delta\psi, & a'\,, & b'\,, & c' \\
\Delta\chi, & a''\,, & b''\,, & c''\,,
\end{array}
$$

since T is supposed to vanish. Hence

$$
\left.\begin{aligned}
\Delta\mathrm{T}_0&=\{-4+\mathrm{n}+2(n-m):(n-1)\}\mathrm{HT}_1\\
&=\{-3n+8+2m-2+2n-2m\}\mathrm{HT}_1:(n-1)\\
&=-\mathrm{HT}_1.
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (14)
$$

So that, substituting in (12), we finally find

$$
\left.\begin{aligned}
(t:m)\Delta\mathrm{T}&=2\mathrm{m}\{1+(\tau-1):(n-1)\}\mathrm{HT}_1-\mathrm{HT}_1\\
&=\mathrm{m}\{1+2(\tau-1):(n-1)\}\mathrm{HT}_1+(n-2m)\mathrm{HT}_1:m.
\end{aligned}\right\}\quad . \quad . \quad . \quad (15)
$$

If the degree of U be double that of $\phi$, $\psi$, . . , *i. e.* if $n=2m$, the last term of this expression will vanish, and $\mathrm{m}=1$; and the equation (15) will be identical with the equation for determining the two branches of three-point contact of T with U at a principal point of the system. But (15) has been formed on the supposition that $\varpi:\varpi'$ satisfies the equation $\Delta\mathrm{V}=\mathrm{n}\theta\mathrm{H}$, viz. the equation for determining the two branches of thre-epoint contact of V with U, at the same point. Hence, in the case where $n=2m$, and at the principal points of the system, the branches of three-point contact of T and U coincide with those of V with U.

It is moreover clear that a similar process may be applied to the functions $\mathrm{Q}z-\mathrm{R}y$, . ., since they are all of the form $\mathrm{A}+\mathrm{mUA}_1$, . ., and that similar results will be obtained. And a transformation similar to that adopted in § 1 will show that the same theorem holds good for the surfaces P, Q, R, S, as for the surface T.

It follows also, as in § 1, that the parabolic points of P, Q, R, S, T coincide with those of U; and also that when $m=n$, the parabolic points of the Hyperjacobian surfaces generally coincide with the principal points of the system.

It is perhaps worth while to calculate the Hessian of the Hyperjacobians at the principal points of the system. And it will be observed that the following calculation applies to all functions which can be expressed in the form $(t:m)\mathrm{P}=\mathrm{P}_0+\mathrm{mUP}_1$. In the first place, forming the Hessian (say $\mathrm{H}_0$) of the left-hand side of this equation, and writing down only the first lines of the determinants, we find

$$
\begin{aligned}
\mathrm{H}_0(\mathrm{P}t:m)&=(t:m)^4\ \partial_x^2\mathrm{P},\ \partial_x\partial_y\mathrm{P},\ \partial_x\partial_z\mathrm{P},\ \partial_x\partial_t\mathrm{P}+\partial_x\mathrm{P}:t\\
&=(p:t)(t:m)^4\,\partial_x^2\mathrm{P},\ \partial_x\partial_y\mathrm{P},\ \partial_x\partial_z\mathrm{P},\ \partial_x^2\mathrm{P}\\
&=p(p-1)^{-1}(t:m)^4\,\partial_x^2\mathrm{P},\ \partial_x\partial_y\mathrm{P},\ \partial_x\partial_z\mathrm{P},\ \partial_x\partial_t\mathrm{P}\\
&=p(p-1)^{-1}(t:m)^4\mathrm{H}_0\mathrm{P}.
\end{aligned}
$$

Again, differentiating the right-hand side of the same equation, we obtain

$$\begin{array}{l} \partial_x^2(\mathrm{P}t:m)\partial_x^2\mathrm{P}_0 \quad +\mathrm{m}\{u_1\mathrm{P}_1+u\partial_x\mathrm{P}_1+u\partial_x\mathrm{P}_1+\mathrm{U}\partial_x^2\mathrm{P}_1 \quad \} \\ \partial_y\partial_x(\mathrm{P}t:m)\partial_y\partial_x\mathrm{P}_0+\mathrm{m}\{w'\mathrm{P}_1+u\partial_y\mathrm{P}_1+v\partial_x\mathrm{P}_1+\mathrm{U}\partial_y\partial_x\mathrm{P}_1\} \\ \quad \vdots \qquad \vdots \qquad \vdots \end{array}$$

Hence

$$\begin{array}{l} \mathrm{m}^{-4}\mathrm{H}_0(\mathrm{P}t:m) \\ = \begin{vmatrix} \partial_x^2\mathrm{P}_0:\mathrm{m}+u_1\mathrm{P}_1+u\partial_x\mathrm{P}_1+u\partial_x\mathrm{P}_1, & \partial_x\partial_y\mathrm{P}_0:\mathrm{m}+w'\mathrm{P}_1+u\partial_x\mathrm{P}_1+u\partial_y\mathrm{P}_1, & \ldots \\ \partial_y\partial_x\mathrm{P}_0:\mathrm{m}+w'\mathrm{P}_1+u\partial_y\mathrm{P}_1+v\partial_x\mathrm{P}_1, & \partial_y^2\mathrm{P}_0:\mathrm{m}+v_1\mathrm{P}_1+v\partial_y\mathrm{P}_1+v\partial_y\mathrm{P}_1, & \ldots \\ \vdots & \vdots & \vdots\;\vdots \end{vmatrix} \end{array}$$

$$= \begin{vmatrix} \partial_x^2\mathrm{P}_0:\mathrm{m}+u_1\mathrm{P}_1, & \partial_x\partial_y\mathrm{P}_0:\mathrm{m}+w'\mathrm{P}_1 & \ldots & \partial_x\mathrm{P}_1, & u \\ \partial_y\partial_x\mathrm{P}_0:\mathrm{m}+w'\mathrm{P}_1 & \partial_y^2\mathrm{P}_0:\mathrm{m}+v_1\mathrm{P}_1 & \ldots & \partial_y\mathrm{P}_1, & v \\ \partial_z\partial_x\mathrm{P}_0:\mathrm{m}+v'\mathrm{P}_1 & \partial_z\partial_y\mathrm{P}_0:\mathrm{m}+u'\mathrm{P}_1 & \ldots & \partial_z\mathrm{P}_1, & w \\ \partial_t\partial_x\mathrm{P}_0:\mathrm{m}+l'\mathrm{P}_1 & \partial_t\partial_y\mathrm{P}_0:\mathrm{m}+m'\mathrm{P}_1 & \ldots & \partial_t\mathrm{P}_1, & k \\ \partial_x\mathrm{P}_1 & \partial_y\mathrm{P}_1 & & . & -1 \\ u & v & \ldots & -1, & 1 \end{vmatrix}$$

But if $p_0$, $p_1$, represent the degrees of $\mathrm{P}_0$, $\mathrm{P}_1$, respectively, we have

$$\begin{array}{l} x(\text{col. }1)+y(\text{col. }2)+z(\text{col. }3)+t(\text{col. }4)-(n-1)\mathrm{P}_1(\text{col. }6) \\ = \begin{vmatrix} (p_0-1)\partial_x\mathrm{P}_0:\mathrm{m}+(n-1)u\mathrm{P}_1-(n-1)u\mathrm{P}_1 = (p_0-1)\partial_x\mathrm{P}_0 \\ (p_0-1)\partial_y\mathrm{P}_0:\mathrm{m}+(n-1)v\mathrm{P}_1-(n-1)v\mathrm{P}_1 \quad (p_0-1)\partial_y\mathrm{P}_0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ (p_1+n-1)\mathrm{P}_1 \qquad\qquad (p_0+n-1)\mathrm{P}_1 \\ 0 \qquad\qquad 0 \end{vmatrix} \end{array}$$

But if P be of the same form as T in this section, or of any form having $u$, $v$, $w$, $k$ as its first four columns, then at the principal points $(\partial_x, \partial_y, \partial_z, \partial_t)\mathrm{P}_0=0$; and consequently all the terms of this column will vanish except the fifth, which will $=(p_0+n-1)\mathrm{P}_1$. Further, if we operate in a similar manner upon the lines, viz. if, for line 6, we write

$$x(\text{line }1)+y(\text{line }2)+z(\text{line }3)+t(\text{line }4)-(n-1)\mathrm{P}_1\text{ line }6,$$

the whole expression will $=(p_1+n-1)^2(n-1)^{-2}\times$

$$\begin{vmatrix} \partial_x^2\mathrm{P}_0:\mathrm{m}+u_1\mathrm{P}_1, & \partial_x\partial_y\mathrm{P}_0:\mathrm{m}+w'\mathrm{P}_1, & \ldots \\ \partial_y\partial_x\mathrm{P}_0:\mathrm{m}+w'\mathrm{P}_1, & \partial_y^2\mathrm{P}_0:\mathrm{m}+v_1\mathrm{P}_1, & \ldots \\ \partial_z\partial_x\mathrm{P}_0:\mathrm{m}+v'\mathrm{P}_1, & \partial_z\partial_y\mathrm{P}_0:\mathrm{m}+u'\mathrm{P}_1, & \ldots \\ \partial_t\partial_x\mathrm{P}_0:\mathrm{m}+l'\mathrm{P}_1, & \partial_t\partial_y\mathrm{P}_0:\mathrm{m}+m'\mathrm{P}_1, & \ldots \end{vmatrix}$$

Similarly, at the points in question, P being of the form indicated above,

$$(\partial_x, \partial_y, \partial_z)^2 P_0 = 0,$$

$$\partial_x\partial_t P_0 : \mathrm{m} = \partial_x P : \mathrm{m} = m P_1 u : t,$$

$$\partial_y\partial_t P_0 : \mathrm{m} = \partial_y P : \mathrm{m} = m P_1 v : t,$$

$$\partial_z\partial_t P_0 : \mathrm{m} = \partial_z P : \mathrm{m} = m P_1 w : t,$$

$$\partial_t^2 P_0 : \mathrm{m} = \partial_t P : \mathrm{m} = m P_1 k : t.$$

Hence, finally, the expression sought $=(p_1+u-1)^2(u-1)^{-2}P_1^4 \times$

$$\begin{array}{llll} u_1 & w' & v' & l'+mu:t, \\ w' & v_1 & u' & m'+mv:t, \\ v' & u' & w_1 & u'+mw:t \\ l'+mu:t & m'+mv:t & n'+mw:t & k_1+mk:t, \end{array}$$

$$=(p_1+n-1)^2(n-1)^{-2}(m+n-1)^2 P_1^4 t^{-2} \times \begin{array}{llll} u_1, & w', & v', & u', \\ w', & v_1, & u', & v', \\ v', & u', & w_1, & w', \\ u', & v', & w', & k_1 \end{array}$$

$$=(p_1+n-1)^2(n-1)^{-2}(m+n-1)^2 P_1^4 H_0.$$

That is to say, at the principal points the Hessian of P vanishes with that of U.

## § 4. *Hyperjacobian Surfaces and Curve of a four-fold Pencil.*

Consider as before the surface U, and the four-fold pencil

$$V = \mathrm{a}\phi + \mathrm{b}\psi + \mathrm{c}\chi + \mathrm{d}w + \mathrm{e}\varrho = 0. \quad \ldots \ldots \ldots \quad (1)$$

If we now form the conditions for three-branch contact between U and V, we shall be able to eliminate the ratios a : b : . . . in two different ways, and thus deduce as before the Hyperjacobian curve. In order to form the expressions required, it will be convenient to write

$$\left.\begin{array}{lllll} \Delta & =\varpi'^2\Delta_{00} & -2\varpi'\varpi\Delta_{01} & +\varpi^2\Delta_{11}, \\ H & =I\varpi'^2 & -2J\varpi'\varpi & +K\varpi^2, \\ \Delta\phi & =L\varpi'^2 & -2M\varpi'\varpi & +N\varpi^2, \\ \Delta\psi & =L'\varpi'^2 & -2M'\varpi'\varpi & +N'\varpi^2, \\ \vdots & \vdots & \vdots & \vdots \end{array}\right\} \quad \ldots \ldots \ldots \quad (2)$$

we shall then have for the Hyperjacobian curve the following expression:—

$$\left.\begin{array}{rccccccc} \mathrm{P}, \mathrm{Q}, \ldots = & u, & v, & w, & k, & \mathrm{nI}, & \mathrm{nJ}, & \mathrm{nK}, \\ & a, & b, & c, & d, & \mathrm{L}, & \mathrm{M}, & \mathrm{N}, \\ & a', & b', & c', & d', & \mathrm{L}', & \mathrm{M}', & \mathrm{N}'; \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right\} \quad \ldots \ldots \quad (3)$$

and it is not difficult to see that, by a transformation similar to that used in § 1, we may write

$$(t:m)(\mathrm{P}, \mathrm{Q}, \ldots) = \mathrm{P}_0 + \mathrm{mUP}_1, \mathrm{Q}_0 + \mathrm{mUQ}_1, \ldots \quad \ldots \ldots \quad (4)$$

and consequently that

$$(t:m)(\partial_x, \partial_y, \partial_z, \partial_t)(\mathrm{P}, \mathrm{Q}, \ldots) = \mathrm{m}(u, v, w, k)(\mathrm{P}_1, \mathrm{Q}_1, \ldots); \quad \ldots \quad (5)$$

that is to say, the Hyperjacobian surfaces touch the given surface at the principal points, and that the Hyperjacobian curve has a node at those points.

Again, a transformation similar to that employed in § 3 will give

$$(t:m)\Delta\mathrm{P} = \mathrm{m}\{1+2(m-1):(n-1)\}\mathrm{HP}_1 + (n-2m)\mathrm{HP}_1 : m. \quad \ldots \quad (6)$$

But it is to be observed that a similar process would have led to the relations

$$(t:m)\Delta_{00}\mathrm{P} = \mathrm{m}\{1+2(m-1):(n-1)\}\mathrm{H}_{00}\mathrm{P}_1 + (n-2m)\mathrm{H}_{00}\mathrm{P}_1 : m, \quad \ldots \quad (7)$$

as well as to the corresponding relations with the suffixes 0, 1; 1, 1 respectively. These show that, in the case considered before, viz. where $n=2m$, the Hyperjacobian surfaces have three-branch contact with the given surface, and consequently with one another, at the principal points. At the same points the Hyperjacobian curve will have a triple point.

### § 5. *Nature of the Contact at the Secondary Points of the System.*

We have hitherto considered the degree of the contact of the Hyperjacobian surfaces, and the nature of the points on the Hyperjacobian curve, at the principal points of the system. Suppose that, at some of the principal points, the surfaces U, $\phi$, $\psi$, ... not only meet, but touch one another; and let these points be called the *secondary points* of the system. When this is the case we shall have the relations

$$\left.\begin{array}{l} u : v : w : k \\ = a : b : c : d \\ = a' : b' : c' : d' \\ = \ldots \end{array}\right\} \quad \ldots \ldots \quad (1)$$

Suppose now that P, Q have the same values as in § 4, and that P represents a determinant containing the first four columns; say, let

$$\mathrm{P}, \mathrm{Q}, \mathrm{R} = \overset{*}{u}, \overset{*}{v}, \overset{*}{w}, \overset{*}{k}, \mathrm{nI}, \mathrm{nJ}, \mathrm{nK}. \quad \ldots \ldots \quad (2)$$

It is then, in the first place, clear that in virtue of these relations we may, at the points in question, regard any two of the first four columns of the determinant P as, à une facteur près, equal to one another; and, consequently, any derivative of P in which any two of those first four columns remain unaffected will *ipso facto* vanish.

Hence, for our present purpose, we shall have

$$\mathrm{P}=0,\ (\partial_x, \partial_y, \partial_z, \partial_t)\mathrm{P}=0,\ \Delta\mathrm{P}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, if we write $(\partial_x\Delta)$ for $(\partial_x\mathrm{A}, \ldots)(\partial_x, \partial_y, \partial_z, \partial_t)^2$, then

$$\partial_x\Delta\mathrm{P}=(\partial_x\Delta)\mathrm{P}+\Delta\partial_x\mathrm{P}.$$

But $(\partial_x\Delta)\mathrm{P}$ vanishes for the same reason as $\Delta\mathrm{P}$; hence operating upon the first derivative of the equation $(t:m)\mathrm{P}=\mathrm{P}_0+\mathrm{mUP}_1$, viz. upon the equation

$$(t:m)\partial_x\mathrm{P}=\partial_x\mathrm{P}_0+\mathrm{m}u\mathrm{P}_1+\mathrm{mU}\partial_x\mathrm{P}_1,$$

and putting $p, p_1, p', p_1'$ for the degrees of $\mathrm{P}, \mathrm{P}_1, \Delta\mathrm{P}, \Delta\mathrm{P}_1$ respectively, we shall obtain

$$\left.\begin{aligned}(t:m)\partial_x\Delta\mathrm{P}=\Delta\partial_x\mathrm{P}_0+\mathrm{m}\{\mathrm{P}_1\Delta u+u\Delta\mathrm{P}_1+2\mathrm{H}\partial_x\mathrm{P}_1+4\mathrm{H}\partial_x\mathrm{P}_1\\ +\mathrm{U}\Delta\partial_x\mathrm{P}_1+2(p_1-1)\mathrm{H}\partial_x\mathrm{P}_1;(n-1)\}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

But since $\mathrm{P}_0$ contains the columns $\mathrm{U}, u, v, w$, it follows that $\partial_x\Delta\mathrm{P}_0$ must contain either the column U, which vanishes, or two of the columns $u, v, w$, any two of which have been shown to be, à une facteur près, identical. Hence

$$(\Delta\partial_x, \Delta\partial_y, \Delta\partial_z, \Delta\partial_t)\mathrm{P}_0=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Also, since $\mathrm{P}_1$ contains the three columns $u, v, w$, it follows that

$$(\partial_x, \partial_y, \partial_z, \partial_t)\mathrm{P}_1=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and we may therefore conclude from (4) that in the present case

$$\partial_x\Delta\mathrm{P}:u=\partial_y\Delta\mathrm{P}:v=\partial_z\Delta\mathrm{P}:w=\partial_t\Delta\mathrm{P}:k=m\mathrm{m}\Delta\mathrm{P}_1:t; \quad . \quad . \quad . \quad . \quad (7)$$

that is to say, at the secondary points of the system the Hyperjacobian surfaces have four-branch contact with the given surface, and consequently with one another; and the Hyperjacobian curve has a quadruple point at these points.

It does not, however, appear that the contact between the Hyperjacobian surface and the given surface is more than four-branched. This will be seen from the following process, which, although leading to only a negative result, is perhaps worth placing on record on account of the peculiarity of the algebraical result.

Operating with $\Delta$ upon the equation $(t:m)\mathrm{P}=\mathrm{P}_0+\mathrm{mUP}_1$, we obtain

$$\begin{aligned}(t:m)\Delta\mathrm{P}+2\mathrm{D}'\mathrm{P}:m&=\Delta\mathrm{P}_0+4\mathrm{mHP}_1+\mathrm{mU}\Delta\mathrm{P}_1+2\mathrm{m}p_1\mathrm{HP}_1:(n-1)\\ &=\Delta\mathrm{P}_0+2\mathrm{m}\{2+p_1:(n-1)\}\mathrm{HP}_1+\mathrm{mU}\Delta\mathrm{P}_1.\end{aligned}$$

Operating a second time:

$$(t:m)\Delta^2 P+2D'\Delta P:m+2(\Delta D')P:m+2D'\Delta P:m+4(A,\ldots)(\partial_x D',\ldots)(\partial_x P,\ldots):m$$
$$=\Delta^2 P_0+2m\{2+p_1:(n-1)\}\{P_1\Delta H+H\Delta P_1+2(A,\ldots)(\partial_x H,\ldots)(\partial_x P_1,\ldots)\}$$
$$+m\{4H\Delta P_1+U\Delta^2 P_1+2p_1'H\Delta P_1:(n-1)\}.$$

But, in virtue of (3) and (6),

$$(\Delta D)P=0,\ (A,\ldots)(\partial_x H,\ldots)(\partial_x P_1,\ldots)=0,\ (A,\ldots)(\partial_x D',\ldots)(\partial_x P,\ldots)=0; \quad (8)$$

and consequently

$$(t:m)\Delta^2 P+4mH\Delta P_1:(n-1)=\Delta^2 P_0+2m\{4+(p_1+p_1'):(n-1)\}H\Delta P_1. \quad (9)$$

Again (writing down only the first lines of the determinants), since

$$\Delta P_0=-HP_1+U,\ \Delta(u,\ldots), \quad (10)$$

$$\begin{aligned}\Delta^2 P_0=-H\Delta P_1-P_1\Delta H-2(A,\ldots)(\partial_x H,\ldots)(\partial_x P_1,\ldots)&\\ +4H,\ \Delta(u,\ldots)+U,\ \Delta^2(u,\ldots)+2u,\ A'\Delta(u,\ldots)&\\ +2v,\ B'\Delta(u,\ldots)&\\ +2w,\ C'\Delta(u,\ldots)&\\ +2k,\ D'\Delta(u,\ldots)&;\end{aligned}$$

or omitting terms which vanish,

$$\begin{aligned}\Delta^2 P_0=-H\Delta P_1+4H,\ \Delta(u,\ldots)-8H,\ \Delta(u,\ldots)+2A'\{u,\ \Delta(u,\ldots)\}&\\ +2B'\{v,\ \Delta(u,\ldots)\}&\\ +2C'\{w,\ \Delta(u,\ldots)\}&\\ +2D'\{k,\ \Delta(u,\ldots)\}&.\end{aligned}$$

But

$$u,\ \Delta(u,\ldots)=\Delta(u,u,\ldots)-\Delta u,\ u,\ldots-2(A,\ldots)(u_1,w',v')(\partial_x,\ldots)(u,\ldots);$$

and of these terms $\Delta(u,u,\ldots)$ vanishes identically, and $\Delta u, u, \ldots$ contains the three columns $u, v, w$; so that $A'\{\Delta u, u, \ldots\}$ will contain two of them, and will consequently vanish.

But if we retain only terms which contain not more than two of the columns $u, v, w, k$, and which, after the operations $A'$, $B'$, $C'$, $D'$, will consequently contain only one such column, we shall have

$$\begin{aligned}u,\ \Delta(u,\ldots)=2u,\ A'u,\ w',\ w,\ldots&+2u,\ A'u,\ v,\ v',\ldots\\ +2u,\ B'u,\ v_1,\ w,\ldots&+2u,\ B'u,\ v,\ u',\ldots\\ +\ldots\qquad&+\ldots\end{aligned}$$

and consequently

$$\begin{aligned}A'\{u,\ \Delta(u,\ldots)\}=2u,\ A'u,\ w',\ A'w,\ldots&+2u,\ A'u,\ A'v,\ v',\ldots\\ +2u,\ B'u,\ v_1,\ A'w,\ldots&+2u,\ B'u,\ A'v,\ u',\ldots\\ +\ldots\qquad&+\ldots\end{aligned}$$

But since col. $w=\lambda$ col. $u$, it follows that

$$\mathrm{A}' \text{ col. } w = \text{col. } u \,.\, \mathrm{A}'\lambda + \lambda\, \mathrm{A}' \text{ col. } u\,;$$

so that the expression $\mathrm{A}'\{u, \Delta(u, \ldots)\}$ will vanish. The same will obviously be the case with the results of $\mathrm{B}'_1$ and $\mathrm{C}'$ on the same expression; and we may, in fact, conclude as follows:—

$$\begin{aligned}
\mathrm{A}'\{u, \Delta(u, \ldots)\} &= 0,\\
\mathrm{B}'\{u, \Delta(u, \ldots)\} &= 0,\\
\mathrm{C}'\{u, \Delta(u, \ldots)\} &= 0,\\
\mathrm{D}'\{u, \Delta(u, \ldots)\} &= \mathrm{D}'\Delta\mathrm{P}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
4\mathrm{H}, \Delta(u, \ldots) &= \{4\mathrm{H}, \Delta(u, v, w), \mathrm{nI}, \mathrm{nJ}, \mathrm{nK}\}\\
&= (4-\mathrm{n})\mathrm{H}\{\Delta(a, b, c), \mathrm{L}, \mathrm{M}, \mathrm{N}\}\\
&= (4-\mathrm{n})\mathrm{H}\Delta\{a, b, c, \mathrm{L}, \mathrm{M}, \mathrm{N}\}\\
&= (4-\mathrm{n})\mathrm{H}\Delta\mathrm{P}_1.
\end{aligned}$$

Hence, finally,

$$\left.\begin{aligned}
\Delta^2\mathrm{P}_0 &= \{-1-4+\mathrm{n}+2m\mathrm{m} : (n-1)\}\mathrm{H}\Delta\mathrm{P}_1\\
&= \{-1+(2m-2-3n+3+2n-2m) : (n-1)\}\mathrm{H}\Delta\mathrm{P}_1\\
&= -2\mathrm{H}\Delta\mathrm{P}_1.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

Collecting the various terms, we find

$$(t : m)\Delta^2\mathrm{P} = 2\{-1+\mathrm{m}[4+(p_1+p'_1-2) : (n-1)]\}\mathrm{H}\Delta\mathrm{P}_1.$$

But since

$$\begin{aligned}
p' &= \text{degree of } \Delta\mathrm{P}, & \therefore p' &= 2n+p-6,\\
p_1 &= \text{degree of } \mathrm{P}_1, & p_1 &= -n+p+1,\\
p'_1 &= \text{degree of } \Delta\mathrm{P}_1, & p'_1 &= 2n+p_1-6,
\end{aligned}$$

consequently

$$\begin{aligned}
&p_1+p'_1 = 2p-4,\\
&p_1+p'_1-2+4(n-1) = 2p'+2,\\
&(4p'+4)(n-m) : m(n-1)-2,\\
&\qquad = \{(4p'+4)(n-m)-2m(n-1)\} : m(n-1)\\
&\qquad = \{[4p'+4-(2n-2)](n-m)+2(n-1)(n-2m)\} : m(n-1)\\
&\qquad = \{[1+2(p'-1) : (n-1)]+[1+2(p-1) : (n-1)]\}\mathrm{m} = 2(n-2m) : m.
\end{aligned}$$

Hence, in the case considered before, viz. where $n=2m$,

$$(t : m)\Delta^2\mathrm{P} = (\mathrm{n}'+\mathrm{n})\mathrm{H}\Delta\mathrm{P}_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $\mathrm{n}'=1+2(p'-1) : (n-1)$; *i. e.* $\mathrm{n}'$ has the same relation to $\Delta\mathrm{P}$ that $\mathrm{n}$ has to P. But since n can in no case vanish, it follows that the Hyperjacobian surfaces of a four-fold pencil cannot in general have more than four-branch contact with the given surface at the secondary points of the system.